

KOSZUL FILTRATIONS AND FINITE LATTICES

DANCHENG LU AND KE ZHANG

ABSTRACT. In this note, we characterize when a finite lattice is distributive in terms of the existences of some particular classes of Koszul filtrations.

1. INTRODUCTION

Let K be a field and L a finite lattice. We use $K[L]$ to denote the polynomial ring over a field K whose variables are elements of L . A binomial in $K[L]$ of the form $ab - (a \vee b)(a \wedge b)$, where $a, b \in L$ are incomparable, is called a basic binomial or a Hibi relation. By definition the *join-meet ideal* I_L is the ideal of $K[L]$ generated by all basic binomials. Set $H[L] := K[L]/I_L$. It was shown in [6] that $H[L]$ is a toric ring if and only if L is distributive. In this case, $H[L]$ is called a *Hibi ring*.

Assume that R is a standard graded K -algebra. Recall that a collection \mathcal{F} of ideals of R is a *Koszul filtration* if

- (1) Every ideal in \mathcal{F} is generated by linear forms,
- (2) The ideals 0 and the maximal graded ideal \mathfrak{m} of R belong to \mathcal{F} ,
- (3) For any ideal $0 \neq I \in \mathcal{F}$, there exists an ideal $J \in \mathcal{F}$ such that $J \subset I$, I/J is cyclic and $J : I \in \mathcal{F}$.

This notion, firstly introduced in [1], was inspired by the work of Herzog, Hibi and Restuccia [5] on strongly Koszul algebras. Its significance is that if R admits a Koszul filtration then R is Koszul, that is, the residue field K has a linear R -free resolution as an R -module, thus it provides an effective way to show a standard graded algebra to be Koszul.

The K -algebra $H[L]$ is standard graded by setting $\deg(a) = 1$ for each $a \in L$. Recall that a subset J of a lattice L is called a *poset ideal* if for any $a, b \in L$ with $a \leq b$ and $b \in J$ one has $a \in J$. Let J be a poset ideal of L . We denote by (\overline{J}) the ideal of $H[L]$ generated by elements $\overline{a} =: a + I_L$ with $a \in J$. The ideal of the form (\overline{J}) is called a *poset ideal* of $H[L]$. It was proved in [3] that if L is distributive then all the poset ideals of $H[L]$ form a Koszul filtration of $H[L]$.

The objective of this note is to characterize when a finite lattice is distributive by the existences of some particular classes of Koszul filtrations. We firstly show that L is distributive if $H[L]$ admits a Koszul filtration in which every element is a poset ideal. Next we introduce the notion of a combinatorial Koszul filtration. By definition, a Koszul filtration \mathcal{F} of $H[L]$ is *combinatorial* if every ideal of \mathcal{F} is generated by residue classes of some elements of L (i.e., variables). We show that a modular lattice L is distributive if and only if $H[L]$ admits a combinatorial Koszul

2010 *Mathematics Subject Classification.* Primary 05E40, 13A02; Secondary 06D50.

Key words and phrases. Koszul filtration, finite lattice, modular, distributive.

filtration. An example (Example 2.2) is given to show that the restrictive attribute “modular” cannot be removed from the last statement.

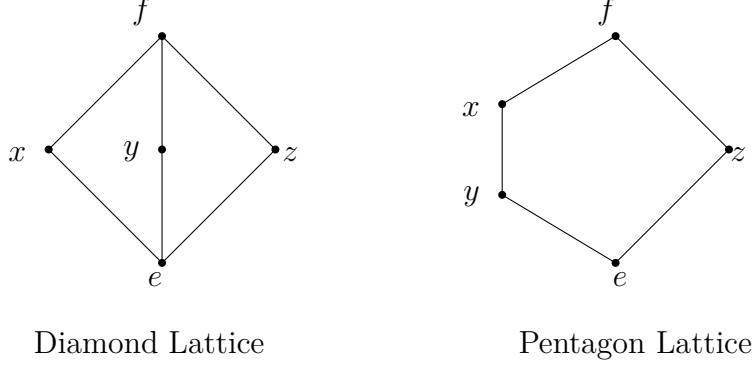


FIGURE 1.

2. CHARACTERIZATIONS OF DISTRIBUTIVE LATTICES

We refer readers to [7] for basic knowledges on finite lattices. For a finite lattice L , we denote by $\max L$ and $\min L$ its largest and least elements respectively. A finite lattice L is called *modular* if $x \leq b$ implies $x \vee (a \wedge b) = (x \vee a) \wedge b$ for all $x, a, b \in L$. A finite lattice is modular if and only if no sublattice of L is isomorphic to the pentagon lattice of Figure 1. Here a nonempty subset L' of L is called a *sublattice* of L if for any $a, b \in L'$, both $a \vee b$ and $a \wedge b$ belong to L' . A finite lattice L is called *distributive* if, for all $x, y, z \in L$, the distributive laws $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ and $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ hold. Every distributive lattice is modular. A modular lattice is distributive if and only if no sublattice of L is isomorphic to the diamond lattice in Figure 1.

Theorem 2.1. *Let L be a finite lattice and $R = H[L]$. The following statements are equivalent:*

- (1) L is distributive;
- (2) All poset ideals of R form a Koszul filtration of R ;
- (3) R admits a Koszul filtration in which every ideal is a poset ideal.

Proof. (1) \Rightarrow (2) It follows from [3, Corollary 2.6].

(2) \Rightarrow (3) Trivially.

(3) \Rightarrow (1) Suppose that L is not distributive. Then L admits a sublattice L' which is either a diamond lattice or a pentagon lattice as in Figure 1. Set $e = \min L'$ and $f = \max L'$. We first prove the following claim.

Claim: If I is a poset ideal of L and e is a maximal element of I , then $(\overline{J}) : \bar{e}$ is not generated by linear forms. Here $J := I \setminus \{e\}$.

In order to prove our claim, we first show that $[(I_L, J) : e]_1$, the linear part of the colon ideal $(I_L, J) : e$, is the K -span of H in $K[L]$, where H is the poset ideal $\{a \in L : a \not\geq e\}$. Let $a \in H$. If $a < e$ then $a \in J$; if a, e are incomparable

then $ae = ae - (a \wedge e)(a \vee e) + (a \wedge e)(a \vee e) \in (I_L, J)$. Hence $H \subseteq [(I_L, J) : e]_1$. For the converse, let $d \in [(I_L, J) : e]_1$. Note that $(I_L, J) = A_1 + A_2 + A_3$, where $A_1 = (J, eH)$, A_2 is the ideal of $H[L]$ generated by binomials $ab - (a \vee b)(a \wedge b)$ with $e \notin \{a, b, a \vee b, a \wedge b\}$ and A_3 is the ideal of $H[L]$ generated by binomials $ab - e(a \vee b)$ with $a \wedge b = e$. Since $ed \in (I_L, J)$, there is a decomposition $ed = d_1 + d_2 + d_3$ such that $d_i \in A_i$ and $\deg(d_i) = 2$ for $i = 1, 2, 3$. For each i , write d_i uniquely as $d_i = eg_i + h_i$, such that every monomial in the support of h_i is not divided by e . Since A_1 is a monomial ideal, $eg_1 \in A_1$ and $h_1 \in A_1$, and this implies $g_1, h_1 \in (J, H) = (H)$. Since $eg_2 + h_2 \in A_2$, there exist a positive integer ℓ and $k_1, \dots, k_\ell \in K$ such that

$$eg_2 + h_2 = k_1(a_1b_1 - (a_1 \wedge b_1)(a_1 \vee b_1)) + \dots + k_\ell(a_\ell b_\ell - (a_\ell \wedge b_\ell)(a_\ell \vee b_\ell)). \quad (1)$$

Here $e \notin \{a_j, b_j, a_j \wedge b_j, a_j \vee b_j\}$ for $j = 1, \dots, \ell$. It follows from (1) that $g_2 = 0$ and $h_2 \in A_2$. Since $eg_3 + h_3 \in A_3$, there exist a positive integer l and $k'_1, \dots, k'_l \in K$ such that

$$eg_3 + h_3 = k'_1(a_1b_1 - e(a_1 \vee b_1)) + \dots + k'_l(a_lb_l - e(a_l \vee b_l)). \quad (2)$$

Here $a_j \wedge b_j = e$ for $j = 1, \dots, l$. It follows from (2) that $h_3 = k'_1a_1b_1 + \dots + k'_la_lb_l$. Note that $h_1 + h_2 + h_3 = 0$ and the support of h_3 is disjoint with the support of h_i for $i = 1, 2$, one has $h_3 = 0$ and this implies $k'_j = 0$ for $j = 1, \dots, l$. In particular, $g_3 = 0$. Therefore, $d = g_1$ belongs to the K -span of H in $K[L]$.

Assume now on the contrary that $(\overline{J}) : \overline{e}$ is generated by linear forms. Since $e(fx - fy) \equiv yzx - xzy \equiv 0 \pmod{I_L}$, we obtain $fx - fy \in (I_L, J) : e$. Here, e, f, x, y, z are all elements of L' , see Figure 1. It follows that $fx - fy \in (I_L, H)$ from the assumption together with the conclusion of the preceding paragraph. Thus we can express $fx - fy$ as $fx - fy$

$$= \sum_{i \in A} k_i(a_ib_i - fx) + \sum_{i \in B} k_i(a_ib_i - fy) + \sum_{i \in C} k_i(a_ib_i - (a_i \vee b_i)(a_i \wedge b_i)) + \sum_{i \in D} k_i h_i t_i. \quad (3)$$

Here, $a_i \vee b_i = f$ and $a_i \wedge b_i = x$ for $i \in A$, $a_i \vee b_i = f$ and $a_i \wedge b_i = y$ for $i \in B$, $(a_i \vee b_i)(a_i \wedge b_i) \notin \{fx, fy\}$ for $i \in C$ and $h_i \in H, t_i \in L$ for $i \in D$. Note that $h_j t_j \neq a_i b_i$ if $i \in A$ and $h_j \in H$. Comparing the coefficients of $a_i b_i$ and fx in (3) respectively one has $k_i = 0$ for each $i \in A \cup B$ and $\sum_{i \in A} k_i = -1$, a contradiction. Thus we complete the proof of our claim.

If R has a Koszul filtration \mathcal{F} consisting of poset ideals, then there is a poset ideal, say (\overline{I}) , which is minimal among poset ideals of \mathcal{F} containing \overline{e} . Then e is a maximal element of I and $J = I \setminus \{e\}$ is a poset ideal of L . It follows that (\overline{J}) is a unique poset ideal contained in (\overline{I}) such that $(\overline{I})/(\overline{J})$ is cyclic and $(\overline{J}) \in \mathcal{F}$. However by the claim, $(\overline{J}) : \overline{e} = (\overline{J}) : (\overline{I})$ is not generated by linear forms, a contradiction. \square

We say that a Koszul filtration \mathcal{F} of $H[L]$ is *combinatorial* if every ideal in \mathcal{F} is generated by the residue classes of some elements of L (i.e., variables). It is natural to ask if only for a distributive lattice L , the algebra $H[L]$ has a combinatorial Koszul filtration. This is not the case as shown by the following example.

Example 2.2. Let P be the pentagon lattice as in Figure 1. Then $H[P]$ has the following combinatorial Koszul filtration:

$$(0), (\bar{x}), (\bar{x}, \bar{y}), (\bar{x}, \bar{z}), (\bar{x}, \bar{y}, \bar{z}), (\bar{x}, \bar{y}, \bar{z}, \bar{e}), (\bar{x}, \bar{y}, \bar{z}, \bar{f}), (\bar{x}, \bar{y}, \bar{z}, \bar{e}, \bar{f}).$$

One can check the following equalities by Singular [2]:

$$0 : (\bar{x}) = 0; (\bar{x}) : (\bar{y}) = (\bar{z}, \bar{x}); (\bar{x}) : (\bar{z}) = (\bar{y}, \bar{x}); (\bar{x}, \bar{y}) : (\bar{z}) = (\bar{x}, \bar{y}); (\bar{x}, \bar{y}, \bar{z}) : \bar{e} = (\bar{x}, \bar{y}, \bar{z}, \bar{f}); (\bar{x}, \bar{y}, \bar{z}) : \bar{f} = (\bar{x}, \bar{y}, \bar{z}, \bar{e}); (\bar{x}, \bar{y}, \bar{z}, \bar{e}) : \bar{f} = (\bar{x}, \bar{y}, \bar{z}, \bar{e}).$$

We need to introduce some more notation. A finite lattice is called *pure* if all maximal chains (totally ordered subsets) have the same length. When a finite lattice is pure, the rank of a in L , denoted by $\text{rank}(a)$, is the largest integer r for which there exists a chain of L of the form $a_0 < a_1 < \dots < a_r = a$. If L is modular, then L is pure and the following equality holds for any $p, q \in L$:

$$\text{rank}(p) + \text{rank}(q) = \text{rank}(p \wedge q) + \text{rank}(p \vee q).$$

We record [4, lemma 1.2] in the following lemma for the later use.

Lemma 2.3. *Let L be a modular non-distributive lattice. Then L has a diamond sublattice L' such that $\text{rank}(\max L') - \text{rank}(\min L') = 2$.*

Theorem 2.4. *Let L be a modular lattice. Then L is distributive if and only if $H[L]$ admits a combinatorial Koszul filtration.*

Proof. If L is distributive, then $H[L]$ admits a Koszul filtration consisting of poset ideals by [3, Corollary 2.6], which is certainly combinatorial. Thus the direction “only if” is proved.

Suppose now that L is non-distributive. Then L has a diamond sublattice L' such that $\text{rank}(\max L') - \text{rank}(\min L') = 2$ by Lemma 2.3. Set $e = \min L'$ and $f = \max L'$. Then for any distinct x, y in the open interval (e, f) , they are incomparable and $x \vee y = f$ and $x \wedge y = e$. For convenience, we write $(e, f) = \{x_1, x_2, \dots, x_n\}$ for some $n \geq 3$.

Claim: If S is a subset of L such that $S \cap [e, f] = \emptyset$, then $(\bar{S}) : \bar{x}$ cannot be generated by residue classes of some variables for any $x \in [e, f]$.

We first consider the case when $x \in (e, f)$, say $x = x_1$. Let us show that $x_1 x_j \notin (I_L, S)$ for $j = 2, \dots, n$. In fact, if $x_1 x_j \in (I_L, S)$ for some $j \neq 1$, then there exist $k_i, l_i \in K$ (the ground field) and $t_i \in L$ such that

$$x_1 x_j = \sum_{a_i, b_i} k_i (a_i b_i - (a_i \vee b_i)(a_i \wedge b_i)) + \sum_{s_i \in S} l_i s_i t_i. \quad (4)$$

Here $\{a_i, b_i\}$ ranges through all incomparable pairs of L . Without loss of generality we assume that $a_1 b_1 = x_1 x_j$ and $\{a_i, b_i\} \subseteq (e, f)$ for $i \leq m := n(n-1)/2$ and $\{a_i, b_i\} \not\subseteq (e, f)$ for $i > m$. This implies $(a_i \vee b_i)(a_i \wedge b_i) = ef$ if $1 \leq i \leq m$ and $(a_i \vee b_i)(a_i \wedge b_i) \neq ef$ if $i > m$. Note that $s_j t_j \neq a_i b_i$ if $s_j \in S$ and $i \leq m$. Comparing the coefficients of $a_i b_i$ with $i \leq m$ and ef in (4) respectively, we obtain $k_1 = 1, k_2 = \dots = k_m = 0$ and $k_1 + k_2 + \dots + k_m = 0$, a contradiction. Thus $x_j \notin (I_L, S) : x_1$ for each $j > 1$. But one has $(x_2 - x_3)$ belongs to $(I_L, S) : x_1$ and this implies that $(\bar{S}) : \bar{x}_1$ is not generated by the residue classes of variables.

For the case when $x = e$, we first see that both $e(f + k_1 a_1 + \cdots + k_r a_r)$ and $e(x_j + k_1 b_1 + \cdots + k_v b_v)$ do not belong to (I_L, S) for any positive integers r, v , any $k_i \in K$, $f \neq a_i \in L$, $x_j \neq b_i \in L$ and $j = 1, \dots, n$. This fact can be proved in a similar manner as we prove $x_1 x_j \notin (I_L, S)$ in the preceding paragraph and so we omit its proof. It follows that neither f nor $x_i, i = 1, \dots, n$ appears in the support of any linear polynomial in $(I_L, S) : e$. Thus, if $(\bar{S}) : \bar{e}$ is generated by linear forms, then $f(x_1 - x_2)$ does not belong to $(I_L, S) : e$. This leads to a contradiction, since $ef(x_1 - x_2) \in I_L$. Hence $(\bar{S}) : \bar{e}$ is not generated by linear forms. The final case when $x = f$ can be proved in the same way as in the case when $x = e$. Thus our claim has been proved.

Now assume on the contrary that $H[L]$ admits a combinatorial Koszul filtration \mathcal{F} . Note that \mathcal{F} has a natural partial order induced by inclusion. Let (\bar{T}) be a minimal element in \mathcal{F} satisfying $T \cap [e, f] \neq \emptyset$. Then $T \cap [e, f]$ consists of a single element, say x . Moreover $(\bar{T} \setminus \{x\})$ is a unique element in \mathcal{F} such that $(\bar{T})/(\bar{T} \setminus \{x\})$ is cyclic. It follows that $(\bar{T} \setminus \{x\}) : \bar{x} \in \mathcal{F}$, which is contradicted to our claim. \square

Example 2.5. Let D be the diamond lattice as in Figure 1. Then $H[D]$ admits no combinatorial Koszul filtrations by Theorem 2.4. However $H[D]$ has the following Koszul filtration:

$$(0), (\bar{x}), (\bar{y} - \bar{z}), (\bar{x}, \bar{y}), (\bar{x}, \bar{z}), (\bar{x}, \bar{y}, \bar{z}), (\bar{x}, \bar{y}, \bar{z}, \bar{e}), (\bar{x}, \bar{y}, \bar{z}, \bar{f}), (\bar{x}, \bar{y}, \bar{z}, \bar{e}, \bar{f}).$$

One can check the following equalities by Singular [2]:

$$(0) : (\bar{x}) = (\bar{y} - \bar{z}); (0) : (\bar{y} - \bar{z}) = (\bar{x}); (\bar{x}) : (\bar{y}) = (\bar{x}, \bar{z}); (\bar{x}) : (\bar{z}) = (\bar{x}, \bar{y}); (\bar{x}, \bar{y}) : (\bar{z}) = (\bar{x}, \bar{y}); (\bar{x}, \bar{y}, \bar{z}) : (\bar{e}) = (\bar{x}, \bar{y}, \bar{z}, \bar{f}); (\bar{x}, \bar{y}, \bar{z}) : (\bar{f}) = (\bar{x}, \bar{y}, \bar{z}, \bar{e}); (\bar{x}, \bar{y}, \bar{z}, \bar{e}) : (\bar{f}) = (\bar{x}, \bar{y}, \bar{z}, \bar{e}).$$

It would be of interest to know if $H[L]$ admits a Koszul filtration for any finite lattice L .

Acknowledge: Thank the referee very much for his/her careful reading and interesting comments!

REFERENCES

- [1] A. Conca, N. V. Trung, G. Valla, *Koszul property for points in projective spaces*, Math. Scand. 89 (2001), 201–216.
- [2] W. Decker, G. M. Greuel, G. Pfister, H. Schönemann, SINGULAR 4-1-0 – A computer algebra system for polynomial computations, <http://www.singular.uni-kl.de> (2016).
- [3] V. Ene, J. Herzog, T. Hibi, *Linear flags and Koszul filtrations*, Kyoto J. Math. 55 (2015), 517–530.
- [4] V. Ene, T. Hibi, *The join-meet ideal of a finite lattice*, J. of Commut. Algebra, 5(2) (2013), 209–230.
- [5] J. Herzog, T. Hibi, G. Restuccia, *Strongly Koszul algebras*, Math. Scand. 86 (2000), 161–178.
- [6] T. Hibi, *Distributive lattices, affine semigroup rings and algebras with straightening laws*, in: Commutative Algebra and Combinatorics, edited by M. Nagata and H. Matsumura, Advanced Studies in Pure Math. Vol. 11 (North-Holland, Amsterdam, 1987), 93–109.
- [7] R. P. Stanley, *Enumerative Combinatorics*; Vol. I (Wadsworth and Brooks/Cole, Monterey, CA, 1986).

DANCHENG LU, DEPARTMENT OF MATHEMATICS, SOOCHOW UNIVERSITY, P.R.CHINA
E-mail address: ludancheng@suda.edu.cn

KE ZHANG, DEPARTMENT OF MATHEMATICS, SOOCHOW UNIVERSITY, P.R.CHINA
E-mail address: 319793057@qq.com